RADFORD'S FORMULA FOR BIFROBENIUS ALGEBRAS AND APPLICATIONS

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Abstract. In a biFrobenius algebra H, in particular in the case that H is a finite dimensional Hopf algebra, the antipode $\mathcal{S}: H \to H$ can be decomposed as $\mathcal{S} = {}_t c \circ c_\phi$ where $c_\phi : H \to H^*$ and ${}_t c : H^* \to H$ are the Frobenius and coFrobenius isomorphisms. We use this decomposition to present an easy proof of Radford's formula for $\mathcal{S}^4$. Then, in the case that the map $\mathcal{S}$ satisfies the additional condition that $\mathcal{S} \star \mathrm{id} = \mathrm{id} \star \mathcal{S} = u\varepsilon$, we prove the *trace formula*: $\mathrm{tr}(\mathcal{S}^2) = \varepsilon(t)\phi(1)$. We finish by applying the above results to study the semisimplicity and cosemisimplicity of H.

1. Introduction

The concept of biFrobenius algebra (or simply bF algebra) was introduced by Y. Doi and M. Takeuchi in [2] as a generalization of finite dimensional Hopf algebras. In particular, bF algebras are equipped with a special antimorphism of algebras and coalgebras that plays the role of the antipode for Hopf algebras and is denoted by the letter $\mathcal{S}$. The theory of bF algebras was further developped in [3], [4], [6] and [7]. For the general theory of Hopf algebras we refer the reader to: [15], [18] and [20].

The main purpose of this paper is to begin the study of a class of bF algebras introduced in [7], that we call here the SbF algebras. It consists of bF algebras with the additional condition that the map $\mathcal{S}$ is the convolution inverse of the identity in H.

A crucial role in the study of the semisimplicity and cosemisimplicity of finite dimensional Hopf algebras is played by Radford's formula for $\mathcal{S}^4$. Hence, after briefly recalling the basic definitions of bF and SbF algebras in Section 2, we present in Section 3 a very short proof of Radford's formula for bF algebras. In Section 4 we prove that the usual formula for $\mathrm{tr}(\mathcal{S}^2)$ is valid in the SbF situation. In Section 5 we study the semisimplicity and cosemisimplicity of SbF algebras over fields of characteristic zero.

The following notations will be in force along the paper. If V is a $\Bbbk$–linear space, $f \in V^*$ and $v \in V$ we define the linear transformation $f|v : V \to V$ as: $(f|v)(w) = f(w)v$. If V is finite dimensional, the map $f \otimes v \mapsto f|v : V^* \otimes V \to \mathrm{End}(V)$ is an isomorphism. Frequently, we represent the elements of $\mathrm{End}(V)$ as elements of $V^* \otimes V$ using the above isomorphism. In this case if $T = \sum f_i \otimes v_i \in \mathrm{End}(V)$, then $\mathrm{tr}(T) = \sum f_i(v_i)$.

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2. BiFrobenius algebras

We review here the basic results and definitions of biFrobenius algebras following [2].

Assume that $(H, m, 1)$ is an associative unital $\Bbbk$–algebra and (H, Δ, ε) is a coassociative, counital $\Bbbk$–coalgebra. We use Sweedler's notation for the comultiplication.

We consider $\rightharpoonup: H \otimes H^* \to H^*$ and $\leftharpoonup: H^* \otimes H \to H^*$ –the standard left and right actions of H on H^*– defined as $(x \rightharpoonup \lambda)(y) = \lambda(yx)$ and $(\lambda \leftharpoonup x)(y) = \lambda(xy)$ for $x, y \in H$, $\lambda \in H^*$.

Dually, one can take $\rightharpoonup: H^* \otimes H \to H$ and $\leftharpoonup: H \otimes H^* \to H$ –the left and right action of H^* on H– defined as follows: if $x \in H$, $\lambda \in H^*$: $x \leftharpoonup \lambda = \sum \lambda(x_1)x_2$ and $\lambda \rightharpoonup x = \sum x_1\lambda(x_2)$.

The algebra H is called a Frobenius algebra if there exists an element $\phi \in H^*$ such that $H^* = \phi \leftharpoonup H$ (which implies that $H^* = H \rightharpoonup \phi$).

The coalgebra H is called a coFrobenius coalgebra if there exists an element $t \in H$ such that $H = t \leftharpoonup H^*$ (which implies that $H = H^* \rightharpoonup t$).

If H is Frobenius or coFrobenius, it is finite dimensional.

If H is Frobenius and coFrobenius, $1 \in H$ is a group like element and $\varepsilon : H \to \Bbbk$ is an algebra homomorphism, ϕ and t can be assumed to satisfy the additional conditions: $t \leftharpoonup \phi = 1$, $\phi \leftharpoonup t = \varepsilon$. In that case t is a right integral and ϕ a right cointegral of H respectively. In other words for all $x \in H$, $tx = \varepsilon(x)t$ and $\phi(x)1 = \sum \phi(x_1)x_2$. Notice also that in this situation $\phi(t) = 1$.

In the above context, one can consider the map $\mathcal{S} : H \to H$, $\mathcal{S}(x) = \sum \phi(t_1x)t_2$ that is called the antipode morphism. In particular $S(1) = t \leftharpoonup \phi = 1$ and $\varepsilon\circ\mathcal{S} = \phi \leftharpoonup t = \varepsilon$.

In this situation we call ${}_tc, c_t : H^* \to H$ the $\Bbbk$–linear maps ${}_tc(f) = t \leftharpoonup f$ and $c_t(f) = f \rightharpoonup t$.

Similarly define ${}_\phi c, c_\phi : H \to H^*$ as the $\Bbbk$–linear maps ${}_\phi c(x) = \phi \leftharpoonup x$ and $c_\phi(x) = x \rightharpoonup \phi$.

In this notation $\mathcal{S} = {}_tc\circ c_\phi$ is a linear bijective endomorphism of H. The composition inverse of $\mathcal{S}$ is denoted as $\overline{\mathcal{S}}$.

In [2] –the notations and hypotheses are the ones considered above– the authors define the 7–uple $(H, m, 1, \Delta, \varepsilon, t, \phi)$ to be a biFrobenius algebra if $\mathcal{S}$ is an antiautomorphism of algebras and coalgebras.

In this case $\phi\circ\mathcal{S} = \lambda$ and $\mathcal{S}t = s$ are respectively a left cointegral and a left integral for H. Moreover, one can define the elements $a \in H$ and $\alpha \in H^*$ –the modular element and the modular function– as $\alpha = t \rightharpoonup \phi$ and $a = \phi \rightharpoonup t$.

In this case, a is a group like element, α a morphism of algebras, $\mathcal{S}a = a^{-1}$, $\alpha\circ\mathcal{S} = \alpha^{-1}$, and for all $x \in H$: $sx = \alpha^{-1}(x)s$ and $\lambda(x)a^{-1} = \sum \lambda(x_1)x_2$

Moreover –see [2]– the maps $x \mapsto ax$, $x \mapsto xa : H \to H$ are automorphisms of coalgebras, i.e., for all $x \in H$,

$$\sum (ax)_1 \otimes (ax)_2 = \sum ax_1 \otimes ax_2 \tag{1}$$

and

$$\sum (xa)_1 \otimes (xa)_2 = \sum x_1 a \otimes x_2 a \tag{2}$$

Similarly, the maps $x \mapsto (x \leftharpoonup \alpha)$, $x \mapsto (\alpha \leftharpoonup x) : H \to H$ are morphisms of algebras. In other words for all $x, y \in H$:

$$\sum \alpha((xy)_1)(xy)_2 = \sum \alpha(x_1)\alpha(y_1)x_2y_2 \tag{3}$$

and

$$\sum (xy)_1\alpha((xy)_2) = \sum x_1y_1\alpha(x_2)\alpha(y_2) \tag{4}$$

Using the uniqueness of the integrals one can prove that: $a \rightharpoonup \phi = \lambda$, $\phi(\mathcal{S}t) = \lambda(t) = \phi(s) = \alpha(a) = 1$.
The following formulæ are valid in an arbitrary bF-algebra.

(i) For all $x \in H$, $x = \sum \phi(t_1x)\overline{\mathcal{S}}(t_2)$
(ii) For all $x, y \in H$, $\sum \phi(y_1x)y_2 = \sum \phi(yx_1)\mathcal{S}(x_2)$
(iii) For all $x, y \in H$, $\sum \phi(xy_2)y_1 = \sum \phi(x_2y)\mathcal{S}(x_1)a$
(iv) If we put in (iii) $x = t$ and $y = t$ we have:
  (a) $\mathcal{S}(\alpha \rightharpoonup x)a = \sum \phi(xt_2)t_1$
  (b) $a\overline{\mathcal{S}}(x) = \sum \phi(t_2x)t_1$
(v) If we write in equation (iii) $x = \mathcal{S}(t)$ we have:
$\sum \phi(\mathcal{S}t_1x)t_2 = \overline{\mathcal{S}}^2(\alpha^{-1} \rightharpoonup x)a^{-1}$

For example equation (ii) can be deduced from the fact that $\mathcal{S}$ is an antimorphism of coalgebras as follows.
Form the equality $y = \sum \phi(y\overline{\mathcal{S}}t_2)t_1$, we deduce that

$$\sum y_1x \otimes y_2 = \sum \phi(y\overline{\mathcal{S}}t_3)t_1x \otimes t_2 \tag{5}$$

As $\mathcal{S}(x) = \sum \phi(t_1x)t_2$, we deduce that $\mathcal{S}(x_2) \otimes \mathcal{S}(x_1) = \sum \phi(t_1x)t_2 \otimes t_3$ and

$$\sum yx_1 \otimes \mathcal{S}x_2 = \sum (y\overline{\mathcal{S}}t_3 \otimes t_2)\phi(t_1x) \tag{6}$$

Applying $\phi \otimes \mathrm{id}$ to the equations (5) and (6) we get (ii).
For later use we compute $c_\phi \circ {}_tc : H^* \to H^*$.
If $\gamma \in H^*$ and $x \in H$ we have

$$\begin{aligned}(c_\phi \circ {}_tc)(\gamma)(x) = ({}_tc(\gamma) \rightharpoonup \phi)(x) = \phi(x({}_tc(\gamma))) = \\ \sum \gamma(t_1)\phi(xt_2) = \gamma(\sum \phi(xt_2)t_1) = \gamma(\mathcal{S}(\alpha \rightharpoonup x)a).\end{aligned} \tag{7}$$

The last equality follows from (iv).

For a bF algebra, it is useful to consider the Nakayama and coNakayama automorphisms –see [2] for the situation of bF algebras and [18] for the case of finite dimensional Hopf algebras–. This morphisms are denoted as $\mathcal{N}$, ${}^c\mathcal{N} : H \to H$ and are defined by the following equalities:

$$\phi \leftharpoonup x = \mathcal{N}(x) \rightharpoonup \phi, \text{ for all } x \in H \tag{8}$$

$$t \leftharpoonup f = (f \circ {}^c\mathcal{N}) \rightharpoonup t, \text{ for all } f \in H^* \tag{9}$$

In more explicit terms $\mathcal{N}$ and ${}^c\mathcal{N}$ can be characterized by the following equations: for all $x, y \in H$, $\phi(xy) = \phi(y\mathcal{N}(x))$ and $\sum {}^c\mathcal{N}(t_2) \otimes t_1 = \sum t_1 \otimes t_2$ if $\Delta(t) = \sum t_1 \otimes t_2$.

It is easy to show that $\mathcal{N}$ is an automorphism of algebras and that ${}^c\mathcal{N}$ is an automorphism of coalgebras.

If we apply ${}^c\mathcal{N}^{-1}$ to the equality (iv) we obtain that ${}^c\mathcal{N}^{-1}(a\overline{\mathcal{S}}(x)) = \sum \phi(t_2 x)\, {}^c\mathcal{N}^{-1}(t_1) = \sum \phi(t_1 x) t_2$.

Hence, we conclude that:

$${}^c\mathcal{N}(x) = a\overline{\mathcal{S}}^2(x) \tag{10}$$

Similarly using (iv) again, we deduce that $a(\overline{\mathcal{S}}\mathcal{N})(x) = \sum \phi(t_2\mathcal{N}(x))t_1 = \sum \phi(xt_2)t_1 = \mathcal{S}(\alpha \rightharpoonup x)a$.

Hence, we conclude that:

$$\mathcal{N}(x) = a^{-1}\mathcal{S}^2(\alpha \rightharpoonup x)a \tag{11}$$

In particular ${}^c\mathcal{N}(a) = a^2$ and $\mathcal{N}(a) = a$.

Next we compute the traces of $\mathcal{N}$ and ${}^c\mathcal{N}$.

From the equality $\mathcal{S}(x) = \sum \phi(t_1 x)t_2$ we obtain that ${}^c\mathcal{N}\mathcal{S}x = \sum \phi(t_2 x)t_1$ and $\overline{\mathcal{S}}\,{}^c\mathcal{N}\mathcal{S} = \sum \phi \leftharpoonup t_2 \otimes \overline{\mathcal{S}}t_1$.

Then

$$\operatorname{tr}({}^c\mathcal{N}) = \operatorname{tr}(\overline{\mathcal{S}}\,{}^c\mathcal{N}\mathcal{S}) = \sum \phi(t_2\overline{\mathcal{S}}t_1) = (\phi\circ\overline{\mathcal{S}})((\mathrm{id} \star \mathcal{S})(t)) \tag{12}$$

Similarly from $x = \sum \phi(t_1 x)\overline{\mathcal{S}}t_2$ we deduce that $\mathcal{N}x = \sum \phi(xt_1)\overline{\mathcal{S}}t_2$, and then:

$$\operatorname{tr}(\mathcal{N}) = \phi((\overline{\mathcal{S}}t_2)t_1) = (\phi\circ\overline{\mathcal{S}})((\mathcal{S} \star \mathrm{id})(t)) \tag{13}$$

We need one more trace computation.

The equality $x = \sum \phi(t_1 x)\overline{\mathcal{S}}t_2$ can be written as: $\mathrm{id} = \sum \phi \leftharpoonup t_1 \otimes \overline{\mathcal{S}}t_2$.

$$\dim(H) = \operatorname{tr}(\mathrm{id}) = \sum \phi(t_1\overline{\mathcal{S}}t_2) = \phi((\mathrm{id} \star \overline{\mathcal{S}})(t)) \tag{14}$$

Next –following [7]– we define a special family of bF algebras, whose representation theoretical properties can be put under a stricter control than

for the general bF algebras. In [7], the second author of this paper considers the class of biFrobenius algebras satisfying the additional condition that

$$\mathcal{S} \star \mathrm{id} = \mathrm{id} \star \mathcal{S} = u\varepsilon. \tag{15}$$

Clearly not all bF algebras satisfy condition (15) –see for example [2]– and in the mentioned paper [7] using known results on the existence of large Hadamard matrices, a family of bF algebras of arbitrarily large dimension satisfying condition (15) and that are not Hopf algebras is constructed.

In other words, for the objects of this family of bF algebras, the multiplication and comultiplication are not related by the so called pentagonal axiom but only by the weaker condition (15).

It is convenient to give an explicit name to the family of all bF algebras satisfying the condition (15).

Definition 1. *If H is a biFrobenius algebra, we say that H is of type S –or that H is an SbF algebra–, if the map $\mathcal{S}$ is the convolution inverse of the identity in* $\mathrm{End}_{\Bbbk}(H)$.

Observation 1. It is worth noticing that one can construct bF algebras H with the property that the identity map $\mathrm{id} : H \to H$ is convolution invertible, but that are not SbF algebras. In this case the convolution inverse of the identity is not the antipode $\mathcal{S} = \sum \phi(t_1 x)t_2$ of the bF algebra.

An example of the above situation is the following. Let $\Bbbk$ be a field of characteristic different from 2 and H the vector space linearly generated by three elements $H = \langle 1, x, y\rangle$. We endow H with the following structure of bF algebra. The multiplication table of H is characterized by the conditions: $1 \in H$ is the unit element and

$$x^2 = \frac{1}{2}x + \frac{3}{2}y, \quad xy = yx = 2 + \frac{1}{2}x + \frac{1}{2}y, \quad y^2 = \frac{3}{2}x + \frac{1}{2}y.$$

The comultiplication is given as:

$$\Delta(x) = \frac{1}{2}x \otimes x,\ \Delta(y) = \frac{1}{2}y \otimes y.$$

Moreover $1 \in H$ is a group like element and ε is defined as: $\varepsilon(x) = \varepsilon(y) = 2$.

If we take $t = 1 + x + y$ and $\phi(a + bx + cy) = a$ one can show directly that $(H, m, 1, \Delta, \varepsilon, t, \phi)$ is a bF algebra with antipode $\mathcal{S} : H \to H$ given by: $\mathcal{S}(1) = 1, \mathcal{S}(x) = y, \mathcal{S}(y) = x$.

Moreover a direct computation shows that the map $\Sigma : H \to H$ given as

$$\Sigma(1) = 1,\ \Sigma(x) = -\frac{2}{3} - \frac{2}{3}x + 2y,\ \Sigma(y) = -\frac{2}{3} + 2x - \frac{2}{3}y$$

is the convolution inverse of the identity in H.

Notice that the above example is in fact a group like algebra in accordance with the definition of [4]

3. A short proof of Radford's formula

Radford's formula for $\mathcal{S}^4$ in the case of Hopf algebra was first proved in full generality in [17] and with predecessors in [19] and [11]. A more recent proof, that is in the spirit of the one we present below, appears in [18]. Generalizations of the formula from the case of Hopf algebras to other situations, braided Hopf algebras, bF algebras –braided and classical, quasi Hopf algebras, weak Hopf algebras, Hopf algebras over rings, and even for the very general case of finite tensor categories, can be found in the following references: [1], [2], [5],[8], [9], [10] and [16].

For the proof we write $\mathcal{S}^2 = {}_tc\circ(c_\phi\circ_t c)\circ c_\phi$. In other words and more explicitly, we have that:

$$\mathcal{S}^2(x) = \sum((c_\phi\circ_t c)(x \rightharpoonup \phi))(t_1)t_2. \tag{16}$$

Then, from equation (7) we deduce that

$$\mathcal{S}^2(x) = \sum(x \rightharpoonup \phi)(\mathcal{S}(\alpha \rightharpoonup t_1)a)t_2 = \sum \phi((\mathcal{S}t_1)ax)\alpha(t_2)t_3.$$

Moreover:

$$\sum \phi((\mathcal{S}t_1)ax)\alpha(t_2)t_3 = \sum \alpha^{-1}(\phi((\mathcal{S}t_1)ax)\mathcal{S}(t_2))t_3 = \tag{17}$$

$$\sum \alpha^{-1}(\phi(\mathcal{S}(t_1)_2ax)\mathcal{S}(t_1)_1)t_2 = \sum \alpha^{-1}(a\phi(\mathcal{S}t_1ax_2)\overline{\mathcal{S}}(ax_1))t_2$$

where in the last equality we used (iii).

Thus, $\mathcal{S}^2(x) = \sum \phi(\mathcal{S}t_1ax_2)\alpha(x_1)t_2 = \sum \alpha(x_1)\overline{\mathcal{S}}^2(\alpha^{-1} \rightharpoonup (ax_2))a^{-1}$, where this last equality follows from (v).

Moreover,

$$\mathcal{S}^2(x) = \overline{\mathcal{S}}^2(\sum \alpha(x_1)ax_2\alpha^{-1}(x_3)a^{-1}) = \overline{\mathcal{S}}^2(a(\alpha^{-1} \rightharpoonup x \leftharpoonup \alpha)a^{-1})$$

and then

$$\mathcal{S}^4(x) = a(\alpha^{-1} \rightharpoonup x \leftharpoonup \alpha)a^{-1}$$

In particular if $\alpha = \varepsilon$ and $a = 1$, i.e., if H is unimodular and counimodular, the antipode satisfies the equality $\mathcal{S}^4 = \mathrm{id}$.

4. The trace formula

From the equality $\mathcal{S}(x) = \sum \phi(t_1x)t_2$ we obtain that:

$$\mathcal{S}^2(x) = \sum \phi(t_1x)\mathcal{S}t_2 \tag{18}$$

and

$$\mathcal{S}^2 = \sum \phi \leftharpoonup t_1 \otimes \mathcal{S}t_2. \tag{19}$$

Theorem 1. *In the situation above, if H is a bF algebra, then* $\mathrm{tr}(\mathcal{S}^2) = \phi((\mathcal{S}\star\mathrm{id})(t))$. *Moreover, if H is of type S, then* $\mathrm{tr}(\mathcal{S}^2) = \varepsilon(t)\phi(1)$.

Proof. Taking traces in the equality $\mathcal{S}^2 = \sum \phi \leftharpoonup t_1 \otimes \mathcal{S}t_2$, we deduce that $\mathrm{tr}(\mathcal{S}^2) = \sum(\phi \leftharpoonup t_1)(St_2) = \phi((\mathcal{S} \star \mathrm{id})(t))$ □

Observation 2. In [2], the authors present the example that they call B_4, of a bF algebra defined as follows. As an algebra $\mathrm{B}_4 = \Bbbk[X]/(X^4)$. If we call $x = X + (X^4)$ and consider the basis $\mathcal{B} = \{1, x, x^2, x^3\}$, then the coalgebra structure of B_4 is given by the following rules: 1 is a group like element, x and x^2 are primitive elements, and $\Delta(x^3) = 1 \otimes x^3 + x \otimes x^2 + x^2 \otimes x + x^3 \otimes 1$. Moreover ε is given as $\varepsilon(1) = 1, \varepsilon(x) = \varepsilon(x^2) = \varepsilon(x^3) = 0$.

If we call $\mathcal{B}^* = \{1^*, x^*, x^{2*}, x^{3*}\}$ the dual basis of V^*, it is easy to show that $t = x^3$ and $\phi = x^{3*}$.

In this situation $\mathcal{S} = \mathrm{id}$, $\phi(1) = \varepsilon(t) = 0$ and $\mathrm{tr}(\mathcal{S}^2) = \mathrm{tr}(\mathrm{id}) = 4$.

This example shows that the S–condition is crucial for the validity of the trace formula. Observe also that B_4 is not semisimple or cosemisimple, but $\mathcal{S}^2 = \mathrm{id}$, see [7]. For Hopf algebras a classical theorem due to Larson and Radford (see [12] and [13]) guarantees that in the case of characteristic zero, if H is semisimple and cosemisimple, then $\mathcal{S}^2 = \mathrm{id}$.

5. Semisimplicity and cosemisimplicity of bF algebras.

The methods we present in this section are similar to the ones appearing in [14] and [18].

In the case that H is unimodular we deduce from equation (11) that $\mathcal{N}(x) = a^{-1}\mathcal{S}^2(x)a$ and hence the map $\mathcal{N}$ is also a morphism of coalgebras.

If H is counimodular, i.e. if $a = 1$, then ${}^c\mathcal{N} = \overline{\mathcal{S}}^2$ and in this case ${}^c\mathcal{N}$ is also a morphism of algebras.

In the case that H is simultaneously unimodular and counimodular, from Radford's formula we deduce that $\mathcal{S}^4 = \mathrm{id}$ and from the above considerations that: $\mathcal{N} = {}^c\mathcal{N} = \mathcal{S}^2 = \overline{\mathcal{S}}^2$ and $\mathcal{N}^2 = \mathrm{id}$.

In this situation we also have that: $\mathcal{S}t = t$ and $\phi \circ \mathcal{S} = \phi$.

Also: $\sum \mathcal{S}t_2 \otimes \mathcal{S}t_1 = \sum t_1 \otimes t_2$, $\sum t_2 \otimes \mathcal{S}t_1 = \sum \overline{\mathcal{S}}t_1 \otimes t_2$, $\sum \mathcal{N}t_2 \otimes t_1 = \sum t_1 \otimes t_2$ and $\sum \mathcal{S}t_2 \otimes t_1 = \sum \overline{\mathcal{S}}t_1 \otimes t_2$.

Then:

$$(\mathrm{id} \star \overline{\mathcal{S}})(t) = (\overline{\mathcal{S}} \star \mathrm{id})(t) = \sum(\mathcal{S}t_2)t_1 = \sum t_2(\mathcal{S}t_1) = \sum(\overline{\mathcal{S}}t_1)t_2 = \sum t_1(\overline{\mathcal{S}}t_2) \tag{20}$$

Lemma 1. *If H is a unimodular and counimodular biFrobenius algebra of type S, then $\mathrm{tr}(\mathcal{N}) = \phi(1)\varepsilon(t)$. Moreover, in the case that the base field has characteristic zero and $(\mathrm{id} \star \overline{\mathcal{S}})(t) = \varepsilon(t)1$, then $\mathcal{N} = \mathcal{S}^2 = {}^c\mathcal{N} = \mathrm{id}$.*

Proof. We already observed that in this situation $\mathcal{N}^2 = \mathrm{id}$. From the equation (14) we deduce that $\dim(H) = \phi(\varepsilon(t)1) = \varepsilon(t)\phi(1) = \mathrm{tr}(\mathcal{N})$. Hence, as all the eigenvalues of $\mathcal{N}$ are ± 1 and its sum –the trace of $\mathcal{N}$– equals the dimension of H, we conclude that the eigenvalue -1 cannot appear so that $\mathcal{N} = \mathrm{id}$. □

If H is a semisimple biFrobenius algebra, it is easy to show that $\varepsilon(t) \neq 0$ –see for example [7]–. Applying ε to the equality $xt = \alpha(x)t$ that is valid for all $x \in H$, we deduce that $\varepsilon(x)\varepsilon(t) = \alpha(x)\varepsilon(t)$, and then that $\alpha = \varepsilon$. In other words a semisimple bF algebra is counimodular.

Similarly if H is cosemisimple, one can conclude that H is unimodular, i.e., $a = 1$.

The above results can be summarized in the following theorem.

Theorem 2. *Assume that H is a biFrobenius algebra of type S defined over an algebraically closed field of characteristic zero. If $\mathcal{S}^2 = \mathrm{id}$, then H is semisimple and cosemisimple. Conversely, if H is semisimple and cosemisimple, and $\sum(\mathcal{S}t_2)t_1 = \varepsilon(t)1$, then $\mathcal{S}^2 = \mathrm{id}$.*

Proof. If $\mathcal{S}^2 = \mathrm{id}$, then $\mathrm{tr}(\mathcal{S}^2) = \phi(1)\varepsilon(t) = \mathrm{tr}(\mathrm{id}) = \dim(H)$. In this situation, $\phi(1) \neq 0$ and $\varepsilon(t) \neq 0$. It is known that in this case –see for example [3] or [7]– H is semisimple and cosemisimple. The rest of the results follow from Lemma 1. □

Observation 3. It would be interesting to know if –similarly than for the situation of finite dimensional Hopf algebras– the result is true without assuming the hypothesis that $\sum(\mathcal{S}t_2)t_1 = \varepsilon(t)1$.

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